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On the geometry of Lie algebras and Poisson tensors

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Abstract. A geometric programme to analyse the structure of Lie algebras is presented with special emphasis on the geometry of linear Poisson tensors. The notion of decomposable Poisson tensors is introduced and an algorithm to construct all solvable Lie algebras is presented. Poisson–Liouville structures are also introduced to discuss a new class of Lie algebras which include, as a subclass, semi-simple Lie algebras. A decomposition theorem for Poisson tensors is proved for a class of Poisson manifolds including linear ones. Simple Lie algebras are also discussed from this viewpoint and lower-dimensional real Lie algebras are analysed.

1. Introduction

The idea of using linear Poisson brackets to understand the structure of Lie algebras can be traced back to the work of Lie [Li88]. In this spirit there have been some suggestions of pursuing this geometric approach for Lie algebra structures, as in Weinstein's study of Poisson manifolds, as a geometrization of Lie algebras [We83]. More recently, a new approach to the classification of Poisson brackets in low dimensions has been presented in [Gr93, Li92]. The main idea in this approach consists in exploiting the geometrical content of the linear Poisson bi-vector defined by the Lie algebra bracket using a volume element and associating to each Poisson tensor a $(n-2)$-form, n being the dimension of the Lie algebra.

In this paper we will continue this line of thought but the emphasis will again be to use the broader frame provided by the geometry of Poisson brackets to think back on the structure of finite-dimensional Lie algebras. That is, either passing to forms using a volume element or, analysing directly the properties of the bi-vector defining the Poisson brackets; in this paper we will make a systematic geometric approach to Lie algebras and we discuss, among other results, a new algorithm that allows us to construct all solvable Lie algebras. This algorithm can also be specialized to build up all nil-potent Lie algebras and provides an alternative approach to previous results and classification techniques [Tu88, Nd94]. It is also shown that all solvable Lie algebras are decomposable in the sense that their Poisson bi-vector is the sum of compatible Poisson tensors. This notion of decomposability is analysed further in the realm of Poisson–Liouville geometry and a decomposition theorem for the Poisson tensor defined by a Lie algebra structure, is presented. This result puts in evidence a class of Lie algebras which are compatible with a volume form and are called Poisson–Liouville structures. This class includes semi-simple Lie algebras and traceless solvable

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Lie algebras. This result also extends a previous decomposition obtained by Liu and Xu for quadratic brackets on vector spaces [Li92]. Simple Lie algebras are discussed from this geometrical viewpoint introducing further geometrical structures, mainly an invariant metric.

The paper will be organized as follows. Section 2 will be devoted to fix the notation and introduce some general facts on the geometry of linear Poisson tensors as well as the notion of decomposition of a Poisson tensor. Extensions of Poisson tensors are briefly discussed in section 3 as well as the algebra of derivations where the notion of parallel derivations are introduced. These ideas are used to build up an algorithm which provides a complete list of solvable Lie algebras in section 4. The structure of the algorithm allows for an immediate classification of particular subfamilies of solvable Lie algebras. Poisson–Liouville structures are discussed in section 5. A decomposition theorem is discussed and linear Poisson structures are shown to be decomposable. Finally in section 6, semi-simple Lie algebras are discussed from the geometrical point of view and in section 7 low-dimensional real Lie algebras up to dimension 4 are analysed as an example of the use of the ideas and techniques discussed in the paper.

2. The Geometry of Lie algebras and Poisson manifolds

A real finite-dimensional Lie algebra L with Lie bracket $[\cdot,\cdot]$, defines in a natural way a Poisson structure $\{\cdot,\cdot\}_L$ on the dual space L^* of L. The natural identification $L \cong (L^*)^*$, allows one to think of L as a subset of the ring of smooth functions $C^\infty(L^*)$. Choosing a linear basis $\{E_i\}_{i=1}^n$ of L, and identifying them with linear coordinate functions x_i on L^* by means of $x_i(x) = \langle x, E_i\rangle$ for all $x \in L^*$, we will define the fundamental commutation relations on L^* by the expression

$$\{x_i, x_j\}_L = c_{ij}{}^k x_k \tag{1}$$

where $[E_i, E_j] = c_{ij}{}^k E_k$, and $c_{ij}{}^k$ denote the structure constants of the algebra. Intrinsically, the Poisson bracket $\{\cdot,\cdot\}_L$ can be defined on $C^\infty(L^*)$ as follows:

$$\{f, g\}_L(x) = \langle x, [\mathrm{d}f(x), \mathrm{d}g(x)]\rangle \tag{2}$$

where $f, g \in C^\infty(L^*)$ and $x \in L^*$. The Poisson bracket $\{\cdot,\cdot\}_L$ is commonly called a Lie–Poisson bracket and it is associated to a bi-vector field Λ_L on L^* written in linear coordinates x_i as

$$\Lambda_L = c_{ij}{}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \tag{3}$$

or, intrinsically,

$$\Lambda_L(\mathrm{d}f \wedge \mathrm{d}g) = \{f, g\}_L .$$

The Jacobi identity for the Poisson bracket $\{\cdot,\cdot\}_L$ is equivalent to the vanishing of the Schouten bracket [Li77, Tu74, Sc53, Ni55], of Λ_L with itself

$$[\Lambda_L, \Lambda_L] = 0 . \tag{4}$$

The Schouten bracket $[\cdot,\cdot]$ is the unique extension of the Lie bracket of vector fields to the exterior algebra of multivector fields, making it into a graded Lie algebra (the grading in this algebra is given by the ordinary degree as multivectors minus one). Given a multivector V, the linear operator $[V,\cdot]$ defines a derivation on the exterior algebra of multivector fields on L^*, whose degree is the ordinary degree of V. If $V = X \wedge Y$ is a monomial bi-vector, then

$$[V, V] = 2X \wedge Y \wedge [X, Y]. \tag{5}$$

For any given closed 1-form θ on L^*, there is an associated vector field

$$X_\theta = -\Lambda_L(\theta) \tag{6}$$

which is an infinitesimal automorphism of Λ_L, i.e.

$$\mathcal{L}_{X_\theta}\Lambda_L = 0$$

and because of that, they will be called (locally) Poisson in what follows. If $\theta = \mathrm{d}f$, the vector field $X_f = -\Lambda(\mathrm{d}f)$ will be called a Poisson vector field with generating function f. Notice that Poisson vector fields are commonly called Hamiltonian vector fields but the change in the terminology used in this paper will be justified by the notions discussed in section 5. It is clear that

$$[X_f, X_g] = X_{\{f,g\}}. \tag{7}$$

This is proved easily using that $\mathcal{L}_{X_f} g = \{f, g\}_L$ and $\mathcal{L}_{X_f}\Lambda_L = 0$. The Poisson vector fields X_i corresponding to the linear coordinate functions x_i, have the expression

$$X_i = -c_{ij}{}^k x_k \frac{\partial}{\partial x_j} \tag{8}$$

and the Poisson bi-vector can be written as

$$\Lambda_L = X_i \wedge \frac{\partial}{\partial x_i}. \tag{9}$$

Notice that this way of writing Λ_L is, of course, not unique.

The linear space L^* carries an action of $Gl(L)$ (the adjoint of the fundamental representation of the group). Identifying the Lie algebra $\mathrm{End}(L)$ of the automorphism group $Gl(L)$ with the algebra $\mathfrak{gl}(n, \mathbb{R})$ of $n \times n$ matrices, this action will define a linear represention of $\mathfrak{gl}(n, \mathbb{R})$ on the space of linear vector fields on L^*. In fact, given $A \in \mathrm{End}(L)$, with associated matrix $A_i{}^j$, it will have associated the linear vector field on L^*

$$X_A = A^j{}_i x_j \frac{\partial}{\partial x_i}. \tag{10}$$

The Abelian group L^* acts by translations on itself and the corresponding vector fields are given by

$$X_a = a_i \frac{\partial}{\partial x_i} \tag{11}$$

for any $a \in L^*$. Defining the linear operators $C_i = \mathrm{ad}(E_i)$, the matrix associated to it will be given by $(C_i)_j{}^k = c_{ij}{}^k$, $i = 1, \ldots, n$, and the Poisson bi-vector Λ_L in (9) is written again as

$$\Lambda_L = X_{C_i} \wedge \frac{\partial}{\partial x_i}. \tag{12}$$

The vector fields X_{C_i} provide a realization of the adjoint representation of L in terms of vector fields on L^*.

Thus the problem of constructing and classifying Lie algebras structures can be translated into differential geometrical terms as the problem of determining all equivalence classes of bi-vectors Λ of the form

$$\Lambda = \sum_{k=1}^{N} X_{A_k} \wedge X_{a_k} \tag{13}$$

with $A_k \in gl(n,\mathbb{R})$ and $a_k \in \mathbb{R}^n$, such that $[\Lambda,\Lambda]=0$. It is important to remark here that if Λ is a Poisson tensor of the form given by (13), the monomials $X_A \wedge X_a$ do not, in general, define Poisson structures by themselves, i.e. $[X_A \wedge X_a, X_A \wedge X_a] \neq 0$ (see lemma 2).

Definition 1. A Poisson tensor Λ will be called decomposable if $\Lambda = \Lambda_1 + \Lambda_2$, where $[\Lambda_1,\Lambda_1]=0$, $[\Lambda_1,\Lambda_2]=0$ and $[\Lambda_2,\Lambda_2]=0$.

In other words, the Poisson bi-vector Λ is decomposable if it is the sum of two compatible Poisson bi-vectors. Notice that if Λ_i define Lie algebra structures, $i=1,2$, the Lie algebra structure $\Lambda = \Lambda_1 + \Lambda_2$ is not the sum of the corresponding Lie algebras, however, the Poisson bi-vectors defining the structures do add up. It is clear that the decomposition of a linear Poisson tensor is preserved under linear changes of coordinates. A Poisson tensor will be called fully decomposable if $\Lambda = \sum_k \Lambda_k$ where Λ_k are monomial Poisson bi-vectors compatible among themselves, i.e. $[\Lambda_k,\Lambda_j]=0$ for all k,j. Notice again that (9) does not, in general, define a decomposition of the Poisson tensor Λ_L.

A direct sum of Lie algebras provides an obvious example of a decomposable Poisson bi-vector. A decomposition of a Poisson tensor is not unique in general. The following example shows that decomposability of Poisson tensors is more general than the direct sum of Lie algebras. For instance, consider the Poisson bi-vector corresponding to the Lie algebra of the group $SO(3)$,

$$\Lambda_{SO(3)} = \left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right)\wedge\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2}.$$

The Poisson bi-vectors $\Lambda_1 = (x_1\partial/\partial x_2 - x_2\partial/\partial x_1)\wedge\partial/\partial x_3$ and $\Lambda_2 = x_3\partial/\partial x_1 \wedge \partial/\partial x_2$ define a decomposition of the simple algebra $\Lambda_{SO(3)}$. Another decomposition of the same algebra is provided by

$$\Lambda_{SO(3)} = X_{R_i} \wedge \frac{\partial}{\partial x_i}$$

where R_i are the matrices defining the adjoint representation of $SO(3)$. It is also remarkable that decomposable Poisson structures provide examples of bi-Hamiltonian manifolds that play a relevant role in the study of integrable dynamical systems [Mg78]. In the following sections we will find several instances of decompositions of certain Poisson tensors.

The previous construction is a particular instance of a more general situation that can be summarized as follows. Let P be a Poisson manifold with Poisson tensor Λ_P, i.e. Λ_P is a bi-vector such that $[\Lambda_P,\Lambda_P]=0$, and G a Lie group acting on P by infinitesimal automorphisms of Λ_P. If Λ_G is a right-invariant Poisson tensor on G, then $\Lambda_G \times \Lambda_P$ defines a Poisson structure on $G\times P$ which is G-invariant and then, the quotient manifold, $(G\times P)/G \cong P$, inherits a Poisson structure that could be denoted by $\Lambda_{G,P}$. If Λ_P is chosen to be zero, then the Poisson structure on G will induce a Poisson structure on P. In fact if $\Lambda_G = \Lambda^{ab}\xi_a \wedge \xi_b$, with ξ_a a basis of right-invariant vector fields on G, then

$$\Lambda_{G,P} = \Lambda^{ab}\hat{\xi}_a \wedge \hat{\xi}_b = \Lambda^{ab}\xi_a^i\xi_b^j\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}$$

where $\hat{\xi}_a = \xi_a^i\partial/\partial x^i$ represents the vector field on P defined by ξ_a. This construction has been used systematically in [Al93] to describe symplectic Lie algebras and to study quadratic Poisson brackets on vector spaces [Li92].

In the particular case of $G = IGl(L^*)$, $P = L^*$ and the natural action of the inhomogeneous general linear group $IGl(L^*) = L^* \dot{\times} Gl(L^*)$ on L^*, there is an induced Poisson bracket on L^* from any right-invariant Poisson bracket on $IGl(L^*)$. Right-invariant

Poisson structures on $IGl(L^*)$ could be obtained by linear combination of monomials of the following types:

$$\xi_A \wedge \xi_B \qquad \xi_A \wedge \xi_a \qquad \xi_a \wedge \xi_b$$

with ξ_A denoting the generator on $Gl(L^*)$ corresponding to the linear operator A on L^*, and ξ_a denotes the generator on L^* associated to the translation by a. Again, it is worthwhile to point it out that the vanishing of the Schouten bracket of the induced Poisson tensor is automatic if the tensor Λ_G on the group is Poisson.

2.1. *Linear structures on Poisson manifolds*

We will not dwell on general aspects of the geometry of Poisson manifolds (see for instance [We83]). However, we will discuss briefly some aspects of the geometry of Poisson bi-vectors associated to Lie algebras which are enriched notably by the linear structure present in the underlying vector space. This geometry is captured by the Liouville vector field Δ on L^*. This vector field is generated by dilations and in linear coordinates x_i is written as

$$\Delta = x_i \frac{\partial}{\partial x_i} .$$

The vector field Δ defines a derivation on the algebra of tensor fields $\mathcal{T}(L^*)$ on L^* by means of the Lie derivative. A tensor field T on L^* will be said to be homogeneous of degree k if it is an eigenvector of $\mathcal{L}_\Delta$ with eigenvalue k, i.e.

$$\mathcal{L}_\Delta T = kT .$$

The algebra of tensor fields on L^* admits a grading

$$\mathcal{T}(L^*) = \bigoplus_{k\in\mathbb{Z}} \mathcal{T}(L^*)^{(k)}$$

where $\mathcal{T}(L^*)^{(k)}$ denotes the space of homogeneous tensors of degree k. This grading is compatible with the tensor product, hence the tensor algebra acquires a trigrading by finite-dimensional spaces $\mathcal{T}_q^p(L^*)^{(k)}$ made of p-contravariant, q-covariant, homogeneous tensors of degree k.

It is clear that linear Poisson bi-vectors Λ_L are homogeneous of degree -1

$$\mathcal{L}_\Delta \Lambda_L = -\Lambda_L .$$

Similarly, the vector fields X_A of degree 0 are homogeneous because $[\Delta, X_A] = 0$. On the contrary, the vector fields X_a are of degree -1. Notice that the lowest degree of homogeneity for a bi-vector is -2. They will correspond to constant ones, and they will necessarily have the form

$$\Lambda^{(-2)} = \sum_i X_{a_i} \wedge X_{b_i}$$

which is fully decomposable.

Notice that if f, g are homogeneous functions of degree $|f|$ and $|g|$, respectively, then $\{f, g\}$ is of degree $|f| + |g| + |\Lambda|$. Then, if Λ is of degree -2, the subspace of quadratic functions (homogeneous of degree 2) will close a finite-dimensional subalgebra of $C^\infty(L^*)$ with respect to the bracket $\{\cdot, \cdot\}$. If Λ_L is a linear Poisson bracket, i.e. of degree -1, then the linear functions will close a finite-dimensional Lie subalgebra (isomorphic to L) on the ring of smooth functions on L^*. The Poisson tensors of the form $X_A \wedge X_B$ are of degree zero, or equivalently, they define quadratic brackets. It is easy to show that there will not exist a finite-dimensional Lie subalgebra of the Lie algebra of analytic functions on L^* with respect to the Poisson bracket defined by such a tensor.

In the linear situation we are discussing, and more generally, in orientable manifolds, we can choose a volume element on L^*. We will choose a constant volume element on L^*, i.e. a homogeneous n-form Ω of degree n, namely,

$$\mathcal{L}_\Delta \Omega = n\Omega\,.$$

There is a one-dimensional subspace of volume forms satisfying the previous condition, all of them proportional. We will fix one of them, denoted by Ω_0, in what follows.

3. The extension problem and the derivation algebra

The construction and classification of arbitrary Lie algebras is an open problem. The classification of all semi-simple Lie algebras over the fields of complex or real numbers due to Cartan is classical, but the classification of solvable algebras, the other main ingredient in the construction of arbitrary Lie algebras because of Levi's theorem, is still not completely understood. Such classification only exists for low dimensions. Recently, a classification of all complex solvable Lie algebras with an Abelian nil-radical has been found by purely algebraic methods [Nd94]. In this section we will discuss the extension problem for linear Poisson tensors and some general properties of the algebra of derivations. In the following section we will propose an algorithm based on a particular extension mechanism for Poisson bi-vector fields, which provides a complete list of solvable Lie algebras.

3.1. Extensions of Poisson bi-vectors

We will consider the sequence of vector spaces $0 \to V \to E \to W \to 0$, i.e. $W \cong E/V$, and the corresponding sequence for the dual spaces, $0 \to W^* \to E^* \to V^* \to 0$. Once a section σ of the first sequence has been chosen, both a Poisson tensor Λ_W and a Poisson tensor Λ_V can be pushed-forward and pulled-back to E^*, respectively. In fact, the choice of such a section allows us to identify E^* with $W^* \times V^*$, and we will define a Poisson structure Λ_E on E^* just by taking the direct sum $\Lambda_E = \Lambda_W \oplus \Lambda_V$. Notice that Λ_E is decomposable because $[\Lambda_V, \Lambda_W] = 0$.

Given two Poisson manifolds (P_1, Λ_1), (P_2, Λ_2), an extension of Λ_1 by Λ_2 is a Poisson manifold (P, Λ), an injective Poisson map $i\colon P_1 \to P$ and a surjective Poisson map $\pi\colon P \to P_2$. If $P_1 = W^*$, $P_2 = V^*$, an extension of the Poisson tensor Λ_W by the Poisson tensor Λ_V is a Poisson tensor Λ_E on E^* such that the corresponding maps $W^* \to E^*$, and $E^* \to V^*$ are Poisson maps. If Λ_W, Λ_V, Λ_E are linear Poisson tensors, this notion corresponds to the ordinary notion of extensions of the associated Lie algebras. In fact, the notion of extension of Poisson tensors introduced above is equivalent to the notion of an extension of the Lie algebra $C^\infty(P_1)$ by the Lie algebra $C^\infty(P_2)$ of the following form:

$$0 \to C^\infty(P_2) \to C^\infty(P) \to C^\infty(P_1) \to 0 \tag{14}$$

(notice that the sequence above cannot be an exact sequence of Poisson algebras). If we restrict our attention to linear Poisson tensors, the identification of the Lie algebras $V \subset C^\infty(V^*)$, $W \subset C^\infty(W^*)$, shows that the restriction of the short exact sequence (14) to the finite-dimensional subalgebras V, E, W, will define an ordinary extension of Lie algebras.

In particular, for a given Lie algebra L, Levi's theorem states the existence of a short exact sequence $0 \to \mathrm{rad}(L) \to L \to L_{ss} \to 0$, where the radical of L, $\mathrm{rad}(L)$, is a maximal solvable ideal of L, and L_{ss} is semi-simple. This result will be translated in geometrical terms by stating that reconstructing Lie algebra structures amounts to extended Poisson tensors corresponding to semi-simple Lie algebras by Poisson tensors corresponding to solvable Lie algebras. In the following section we will concentrate on the construction of solvable Lie algebras, leaving the discussion of semi-simple Lie algebras to section 6.

The simplest non-trivial extension construction will correspond to semi-direct products. For this matter we will assume that there is a linear action of the Lie algebra W on V by derivations, i.e. there is a Lie algebra homomorphism $\rho\colon W \to \mathrm{Der}(V)$. Linear maps of V as elements in $\mathrm{End}(V)$ will define automatically linear vector fields on V^* as discussed in the previous section, equation (10), thus the image of a vector $\xi \in W$ by ρ can be identified with a linear vector field $X_{\rho(\xi)}$ on V^*. Besides, because $\rho(\xi)$ are derivations of the Lie algebra V, we get that the corresponding linear vector fields $X_{\rho(\xi)}$ will preserve the Poisson tensor Λ_V. If we select a linear basis $\{E_a\}$ on W, we will denote the linear vector field $X_{\rho(E_a)}$ simply by Y_a. We will define the extended Poisson tensor on E^* by the formula

$$\Lambda_E = \Lambda_V + Y_a \wedge \frac{\partial}{\partial y_a} + \Lambda_W \tag{15}$$

where y_a will denote the linear coordinates on W^* defined by E_a. Choosing a linear basis $\{E_i\}$ on V and x_i denoting the corresponding linear coordinates on V^*, we can write

$$\Lambda_E = c_{ij}{}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + d_{ai}{}^j x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_a} + f_{ab}{}^c y_c \frac{\partial}{\partial y_a} \wedge \frac{\partial}{\partial y_b}$$

where $c_{ij}{}^k$ and $f_{ab}{}^c$ denote the structure constants defined by the linear Poisson tensors Λ_V and Λ_W, respectively, and $\rho(E_a)(E_i) = d_{ai}{}^j E_j$. The homomorphism property of the map ρ guarantees that Λ_E is a Poisson tensor. The Poisson tensor Λ_E will be called the semi-direct sum of Λ_V and Λ_W with respect to ρ.

An interesting situation arises if $V = W^*$, and ρ is the coadjoint action of W on W^*, and ρ^* is the coadjoint action of W^* on W, we can construct the following linear tensor on $E^* = W \oplus W^*$:

$$\Lambda_E = c_{ij}{}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + c_{ij}{}^k y^i \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial y^k} + f^{ij}{}_k x_i \frac{\partial}{\partial y^j} \wedge \frac{\partial}{\partial x_k} + f^{ij}{}_k y^k \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j}$$

will define a Lie algebra structure if the structure constants $c_{ij}{}^k$, and $f^{ij}{}_k$ are compatible, in other words, if Λ_E defines on E^* a Lie algebra structure then (W, W^*) will be a Lie bi-algebra. If $f^{ij}{}_k = 0$, it is a cotangent algebra, i.e. it is the Lie algebra corresponding to the Lie group structure on T^*G (G being a Lie group with Lie algebra V).

Let us consider the special situation when V is a subspace of codimension one in E. That means that $W = E/V$ is one-dimensional. We can construct a semi-direct product of a Poisson structure on W^* and a Poisson structure on V^*. Because W^* is one-dimensional, the only Poisson structure on it is zero. Thus, the semi-direct product will be given by a linear Poisson structure Λ_V and a linear map from W into linear vector fields on V^* corresponding to derivations on V, but because W is one-dimensional all we need is to select a linear vector field X_A such that A is a derivation of the Lie algebra V. Then the extended Poisson bi-vector (15) will have the form

$$\Lambda_E = \Lambda_V + X_A \wedge \frac{\partial}{\partial y} \tag{16}$$

where y is a linear coordinate on W^*. In general, the condition for a bi-vector of the form in (16) to be Poisson is stated in the following lemma.

Lemma 1. If V is a codimension one subspace of the vector space E and $A \in \mathrm{End}(V)$, then the bi-vector $\Lambda_E = \Lambda_V + X_A \wedge \partial/\partial y$ is a Poisson bi-vector iff

$$\mathcal{L}_{X_A}\Lambda_V = 0 . \tag{17}$$

Proof. It follows immediately from the vanishing of the Schouten tensor $[\Lambda_E, \Lambda_E] = 0$, because it reduces to $[\Lambda_V, X_A \wedge \partial/\partial y] = [\Lambda_V, X_A] \wedge \partial/\partial y = 0$ and the conclusion follows. □

From a geometrical viewpoint the previous construction, (16), of Poisson bi-vectors corresponding to extensions of Abelian one-dimensional Lie algebras, is distinguished because of the following argument.

Lemma 2. Let A be a linear transformation on a vector space E and $a \in E$ an arbitrary vector. Then, the bi-vector $\Lambda = X_A \wedge X_a$ defines a Poisson structure on E^* iff a is a null eigenvector of A or Λ can be rewritten as $X_B \wedge X_b$ where b is a null eigenvector of B.

Proof. Because of (5), the Jacobi identity $[X_A \wedge X_a, X_A \wedge X_a] = 0$, is equivalent to

$$X_A \wedge X_a \wedge X_{A \cdot a} = 0 .$$

Then, if $A \cdot a = 0$, the bi-vector is Poisson because the previous equation is automatically satisfied.

If $A \cdot a = b \neq 0$, then $X_a \wedge X_b \wedge X_A = 0$, implies that the vector field X_A lies in the linear space spanned by the constant fields X_a and X_b. Notice that a and b are parallel iff a is an eigenvector of A with eigenvalue λ. In such case defining $B = A - \lambda P_a$, where P_a is a projector on the subspace generated by a, we obtain the desired result.

Finally, if a and b are independent, the vector field X_A will be written as

$$X_A = f X_a + g X_b$$

with f, g linear functions on E^*. Then, $g(x) = \langle u, x \rangle$, for some $u \in E$. In such case it is clear that

$$X_A \wedge X_a = \langle u, x \rangle X_b \wedge X_a .$$

The linear vector field $\langle u, x \rangle X_b$ defines a linear map B on E, such that $B \cdot a = 0$ iff $\langle u, a \rangle = 0$. Assume now that $\langle u, b \rangle \neq 0$. Notice that if $\langle u, b \rangle = 0$, then considering the linear vector field $X_B = \langle u, x \rangle X_a$ and the constant vector field X_b the conclusion follows. Then, redefining the vector a as

$$\tilde{a} = a - \frac{\langle u, a \rangle}{\langle u, b \rangle} b$$

we have $X_{\tilde{a}} \wedge X_b = X_a \wedge X_b$, and the linear vector field $\langle u, x \rangle X_{\tilde{a}}$ defines a linear map B on E, such that $B \cdot a = 0$ because $\langle u, \tilde{a} \rangle = 0$. □

Selecting linear coordinates such that X_a is a coordinate vector field, say $\partial/\partial x_n$, our bi-vector field looks like

$$X_A \wedge X_a = \sum_{i=1}^{n-1} A_i{}^j x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_n}$$

where the matrix elements $A_n{}^j$ do not appear on the tensor. Finally, because of lemma 2, $A \cdot e_n = 0$, and we obtain that $A_1{}^n = A_2{}^n = \cdots = A_{n-1}^n = 0$. Thus we can choose A to have the block form

$$A = \left(\begin{array}{c|c} A_0 & 0 \\ \hline 0 & 0 \end{array} \right) . \tag{18}$$

Thus, we have proved that in an n-dimensional linear space E,

Lemma 3. If the bi-vector field $X_A \wedge X_a$ is Poisson, then there exists a subspace $V \subset E$ supplementary to $\langle a \rangle$ such that $A(V) \subset V$, and an adapted linear coordinate system x_i on E^* such that it is written as

$$X_A \wedge X_a = \sum_{i,j=1}^{n-1} A_i{}^j x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_n} .$$

As a consequence, monomial linear Poisson tensors will define automatically semi-direct extensions of Abelian one-dimensional algebras. This is the main content of the following proposition.

Proposition 1. The bi-vector $\Lambda_E = \Lambda_V + X_A \wedge \partial/\partial y$ represents a semi-direct extension of a one-dimensional Abelian Lie algebra by the Lie algebra V if and only if X_A is a derivation of Λ_V.

3.2. *The derivation algebra*

We have seen in the previous discussion that any decomposition of $\Lambda_E = \Lambda_V + X_A \wedge X_a$ on E, where $E^* = V^* \oplus \langle a \rangle$, $A \in \mathrm{End}(V)$, defines an extension of the Abelian one-dimensional Lie algebra generated by the vector a, by the Lie algebra defined on V by Λ_V. Thus we have a new Lie algebra on E associated to the Poisson tensor Λ_E defined by the commutation relations

$$\{x_i, x_j\}_E = c_{ij}{}^k x_k \qquad \{x_i, y\}_E = A_i{}^j x_j .$$

In this sense lemma 1 recasts the derivation property of vector fields defining semi-direct extensions discussed in the previous paragraphs. In fact a linear map $A: V \to V$ defines a derivation with respect to the Lie algebra structure described by the Poisson bi-vector Λ_V on V if and only if (17) is satisfied, i.e. if the linear vector field defined by the linear map is an infinitesimal automorphism of the Poisson structure. These vector fields form themselves an algebra, called the derivation algebra of the corresponding Lie algebra, and will be denoted by $\mathrm{Der}(V)$. Thus there is a correspondence between extensions of the particular kind discussed above and $\mathrm{Der}(V)$. The derivation algebra $\mathrm{Der}(V)$ acts on V, defining in this way the semi-direct product $\mathrm{Hol}(V) = V \dot{\times} \mathrm{Der}(V)$, called the holomorph of V, which in some cases characterizes the Lie algebra V [Sh55].

We notice that we could also consider vector fields such that $\mathcal{L}_{X_a}\Lambda = 0$, obtaining in this way the inhomogenous derivation algebra. Among all derivations there is an invariant subalgebra given by those derivations defined by linear vector fields of the form $X_\alpha = \Lambda(\alpha)$, where $\alpha = \alpha^i \mathrm{d}x_i$ is a 1-form on V^*. For X_α to be linear (i.e. of degree zero), α must be of degree 1, therefore α is going to be exact, $\alpha = \mathrm{d}(\alpha^i x_i)$. Then the vector field X_α is Poisson and is a linear combination of the adjoint vector fields X_{C_i} defined in (8), $X_\alpha = \alpha^i X_{C_i}$. These derivations are called inner derivations and will be denoted by $\mathrm{Int}(V)$.

There is yet another subalgebra of derivations, given by linear vector fields which are in the image of Λ on some open dense submanifold. Let k be the maximum rank of Λ, then $\Lambda^k \neq 0$, but $\Lambda^{k+1} = 0$. Then we will say that a derivation X is parallel if $X \wedge \Lambda^k = 0$ and $\mathcal{L}_X \Lambda = 0$ and will be denoted by $\mathrm{Der}^\pi(V)$. This derivations are on the image of Λ on the open dense submanifold made up by the union of maximal dimension symplectic leaves of the Poisson tensor Λ. They form an invariant subalgebra of the full derivation algebra and contains as an invariant subalgebra the algebra of inner derivations. Typical derivations of this kind are those of the form CX_a where C is a linear Casimir function and $\mathcal{L}_{X_a}\Lambda = 0$

(*we must notice that the vector field X_Λ that will be introduced later on, (29) and (37) will* be of this kind).

Therefore, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{Int}(V) & \xrightarrow{\cong} & \mathrm{Int}(V) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{Der}^\pi(V) & \longrightarrow & \mathrm{Der}(V) & \longrightarrow & \mathrm{Out}^T(V) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow\cong & & \\
0 & \longrightarrow & \mathrm{Out}^\pi(V) & \longrightarrow & \mathrm{Out}(V) & \longrightarrow & \mathrm{Out}^T(V) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $\mathrm{Out}^T(V) = \mathrm{Der}(V)/\mathrm{Der}^\pi(V)$ and $\mathrm{Out}^\pi(V) = \mathrm{Der}^\pi(V)/\mathrm{Int}(V)$. Any derivation can be written, once a splitting of the central vertical sequence has been chosen, as a linear combination of inner derivations and outer derivations.

In the specific cases we are going to consider, it will be clear that elements in $\mathrm{Out}(V)$ which are not parallel, should be searched among vector fields transverse to the leaves of the symplectic foliation of Λ. Let us discuss the special case when symplectic leaves are of dimension equal to the dimension of V, then the only outer derivations are going to be in $\mathrm{Out}^\pi(V)$. As we mentioned before typical parallel derivations will have the form CX_a. However, the existence of elements X_a parallel and satisfying $\mathcal{L}_{X_a}\Lambda = 0$, is not the end of the story because to make up a linear vector field we need a linear Casimir function. We can prove that by making the assumption that $CX_a = \Lambda(\theta)$ with θ a linear 1-form we have $X_a = \Lambda(\theta/C)$ with θ/C a smooth function of degree zero. Thus, X_a cannot be in the image of Λ. If ϕ_a^t is the flow of X_a, $(\phi_a^t)_*\Lambda = \Lambda$ implies that $\phi_a^t(\mathbf{0})$ must be a zero of Λ, therefore, Λ does not have isolated zeros but a full line. We conclude that when Λ is non-degenerate on an open dense submanifold, a linear vector field that is not in the image of Λ but preserves Λ will have an integral curve in the zero set of Λ. If this cannot be the solution curve of a linear vector field then there is no outer parallel derivations. The structure of the derivation algebra will be discussed elsewhere.

An interesting illustration of the previous arguments is provided by the 'book algebra' in $(n+1)$-dimensions. This algebra is defined by the Poisson tensor:

$$\Lambda_{\mathrm{book}} = \left(x_1\frac{\partial}{\partial x_1} + \cdots + x_n\frac{\partial}{\partial x_n}\right)\wedge\frac{\partial}{\partial x_{n+1}} = X_{I_n}\wedge\frac{\partial}{\partial x_{n+1}} \tag{19}$$

where I_n is the unit matrix in n-dimensions. The rank of Λ is 2 except along the x_{n+1}-axis, where is zero. Inner derivations are generated by linear Hamiltonian vector fields. These are $X_i = x_i\partial/\partial x_{n+1}$, $i = 1,\ldots,n$, and $X_{n+1} = \Delta_n$, with Δ_n the Liouville vector field in n-dimensions. The symplectic leaves of the Poisson structure are two-dimensional half-planes defined by an arbitrary vector and the x_{n+1}-axis, i.e. $\mathcal{O}_a = \{\lambda a + \mu e_{n+1} \mid \lambda > 0, \mu \in \mathbb{R}, 0 \neq a \in \mathbb{R}^n\}$, and the points on the x_{n+1}-axis (see figure 1).

It is easy to show that there are no non-constant smooth Casimir functions. Outer parallel linear derivations should be of the form $f(x_1,\ldots,x_n)\partial/\partial x_{n+1}$, with f a linear function, but this implies that f must be zero (no Casimirs). On the contrary there are non-parallel outer derivations. They are generated by all linear vector fields of the form X_A with A a linear map with $A^i_{n+1} = 0 = A^{n+1}_i$, $i = 1,\ldots,n+1$. Thus the algebra of linear derivations is $(n+1)n$-dimensional and the algebra of outer derivations has dimension n^2-1.

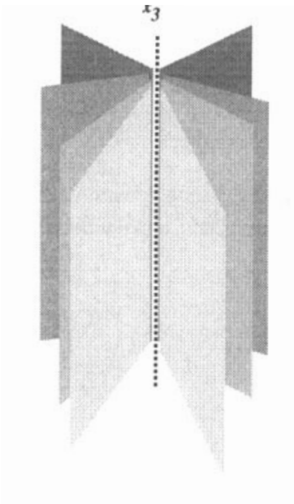

Figure 1. Symplectic leaves of the three-dimensional 'book' algebra.

4. Solvable Lie algebras

The ideas discussed in the previous section can be applied to construct all solvable and all nil-potent Lie algebras in a vector space of arbitrary dimension. Solvable Lie algebras in a vector space of dimension $n+1$ will be constructed using the previously constructed solvable Lie algebras in a vector space of dimension n. In this way we will construct recursively families of solvable (and nil-potent) Lie algebras that exahust all of them. The algorithm works as follows.

Let V_n be a n-dimensional real vector space. The Poisson structure Λ_n^0 is trivial,

$$\Lambda_n^0 = 0 .$$

Let $0 \leqslant j < n$, then if Λ_n^j is the linear Poisson tensor associated to a solvable Lie algebra in V_n^*, and $X_{A_{j+1}}$ is a linear vector field on V_n such that the compatibility condition (17) is satisfied, i.e. $X_{A_{j+1}}$ is a linear derivation of Λ_n^j, then we define the Poisson bi-vector $\Lambda_{n+1}^{j+1}(A_{j+1})$ on $V_{n+1} = V_n \oplus \mathbb{R}$, by

$$\Lambda_{n+1}^{j+1}(A_{j+1}) = \Lambda_n^j + X_{A_{j+1}} \wedge \frac{\partial}{\partial x_{n+1}} .$$

In what follows Λ_{n+1}^{j+1} will be used to denote the family of all linear Poisson bi-vectors $\Lambda_{n+1}^{j+1}(A_{j+1})$. Notice that a Poisson bi-vector $\Lambda_n^j(A_j)$ on V_n constructed using this algorithm will have the form

$$\Lambda_n^j(A_j) = \Lambda_{n-1}^{j-1}(A_{j-1}) + X_{A_j} \wedge \frac{\partial}{\partial x_n} = \cdots = X_{A_1} \wedge \frac{\partial}{\partial x_{n-j+1}} + X_{A_2} \wedge \frac{\partial}{\partial x_{n-j+2}} + \cdots$$
$$+ X_{A_j} \wedge \frac{\partial}{\partial x_n} .$$

The family $A_1, A_2, \ldots, A_j$ are linear maps of the sequence of nested linear spaces $V_{n-j} \subset V_{n-j+1} \subset \cdots \subset V_n$. Because of this structure the linear Poisson tensor $\Lambda_n^j(A_j)$ will be denoted in what follows by $\Lambda_n(A_1, \ldots, A_j)$, and the corresponding Lie algebra by $L_n(A_1, \ldots, A_j)$.

The bi-vector $\Lambda_n(A_1, \ldots, A_j)$ represents a solvable Lie algebra. A simple computation shows that the derived algebra $L_n(A_1, \ldots, A_j)^{(1)} = [L_n(A_1, \ldots, A_j), L_n(A_1, \ldots, A_j)]$ is contained in the Lie algebra $L_{n-1}(A_1, \ldots, A_{j-1})$ defined by $\Lambda_{n-1}(A_1, \ldots, A_{j-1})$. But the

Lie algebra $L_{n-1}(A_1, \dots, A_{j-1})$ is solvable, thus $L_n(A_1, \dots, A_j)^{(1)}$ will be solvable too. In the particular case of $\Lambda_n(A_1)$, then the Lie algebra $L_n(A_1)$ is such that $L_n(A_1)^{(2)} = 0$.

For instance, $\Lambda_n(A_1)$ are monomial bi-vectors of the form described in lemma 3, i e.

$$\Lambda_n(A_1) = X_{A_1} \wedge \frac{\partial}{\partial x_n}$$

and so on. Hence a Poisson tensor on the family Λ_n^j is a decomposable Poisson tensor with j monomial factors. The structure of the families generated by this algorithm is visualized in the following diagram:

$$\begin{array}{ccccccc}
\Lambda_1^0 & & & & & & \\
\downarrow & & & & & & \\
\Lambda_2^1 & \Lambda_2^0 & & & & & \\
\downarrow & \downarrow & & & & & \\
\Lambda_3^2 & \Lambda_3^1 & \Lambda_3^0 & & & & \\
\downarrow & \downarrow & \downarrow & & & & \\
\Lambda_4^3 & \Lambda_4^2 & \Lambda_4^1 & \Lambda_4^0 & & & \\
\downarrow & \downarrow & \downarrow & \downarrow & & & \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \\
\downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\
\Lambda_n^{n-1} & \Lambda_n^{n-2} & \Lambda_n^{n-3} & \dots & \dots & \Lambda_n^1 & \Lambda_n^0 \\
\downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots
\end{array}$$

where vertical arrows indicate the extension construction described in the algorithm. It is important to remark here that because of the nonuniqueness of the decomposition of a Poisson tensor, the families Λ_n^j are not disjoint, in general. However, we can prove the following result.

Proposition 2. If L is a solvable Lie algebra of dimension n with derived series of length l, then L belongs to the family Λ_n^l.

Proof. $n = \dim L$. It is well known ([Ja62], theorem 14, p 52) that for L solvable there exists a chain of solvable ideals L_i, $\dim L_i = i$, such that

$$0 = L_0 \subset L_1 \subset \cdots \subset L_{n-1} \subset L_n = L\,.$$

Each couple $L_i \subset L_{i+1}$ defines a situation similar to the one considered in the previous section, where the Lie algebra structure is an extension of an Abelian one-dimensional algebra L_{i+1}/L_i by the (non-Abelian) L_i. Thus because of proposition 1 the Poisson tensor defined on L^* will be of the form Λ_n^j where j will be the length of the derived series of L. □

Corollary 1. The Poisson tensor Λ_L defined by an arbitrary solvable Lie algebra is decomposable and it can be written as a sum of as many monomial as the length of the derived series of L.

If A is a nil-potent matrix with Λ_n^j representing a nil-potent Lie algebra on $\mathbb{R}^n$, then $\Lambda_{n+1}^{j+1} = \Lambda_n^j + X_A \wedge \partial/\partial x_{n+1}$ represents a nil-potent Lie algebra on $\mathbb{R}^{n+1}$. In fact, notice that in the central series $L(n+1, j+1)^k = [L(n+1, j+1), L(n+1, j+1)^{k-1}]$, $L(n+1, j+1)^0 = L(n+1, j+1)$, the cross terms corresponding to the coordinates $\{x_i, y\} = A_i{}^j x_j$, corresponds to powers of A, thus $L(n+1, j+1)^\infty = 0$.

The classification problem of solvable algebras, can be addressed looking at the structure unveiled by the previous algorithm. Linear isomorphisms preserve the decomposition structure of the Poisson tensors. Thus, Poisson tensors in the diagonal Λ_n^1, $n = 1, 2, \ldots$, will be characterized by a single linear map on a vector space of dimension $n-1$. Normal forms for linear maps are well known and they classify the corresponding solvable Lie algebras. The classification of the diagonal Λ_n^2 brings to the stage the characterization of pairs of non-commuting linear maps on a vector space of dimension n, problem that unfortunately has not been solved yet. In fact, tensors on Λ_n^2 will be of the form

$$\Lambda_n^2 = X_A \wedge \frac{\partial}{\partial x_{n-1}} + X_B \wedge \frac{\partial}{\partial x_n}$$

with A a linear map on $(n-2)$-dimensional subspace, B a linear map on a $(n-1)$-dimensional subspace and satisfying the compatibility condition (17).

4.1. Lower-dimensional solvable Lie algebras

We will illustrate the use of the previous algorithm constructing solvable Lie algebra in low dimensions. The construction starts in dimension 1, where there is a unique trivial (Abelian) Lie algebra $\Lambda_1^0 = 0$.

4.1.1. Dimension 2. In dimension 2 we construct a Lie algebra structure with associated Poisson bi-vector

$$\Lambda_2^1(A) = X_A \wedge \frac{\partial}{\partial x_2} = a x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \qquad a \in \mathbb{R}.$$

The family Λ_2^1 contains the trivial Lie algebra ($a = 0$). If $a \neq 0$ it can be fixed to be 1 after a reparametrization of the coordinate x_2.

It is clear that the tensor field Λ_2^1 is non-degenerate on a dense submanifold ($x_1 \neq 0$) with symplectic leaves the half-planes (open two-dimensional submanifolds) $\mathcal{O}_1 = \{x_1 < 0\}$, $\mathcal{O}_2 = \{x_1 > 0\}$, and the collection of points $\{(0, x_2)\}$ (See figure 2).

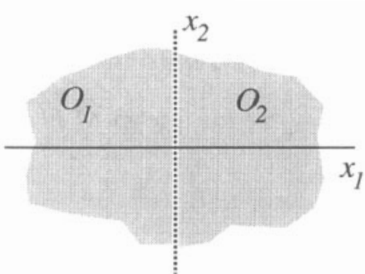

Figure 2. Symplectic leaves of the two-dimensional Poisson tensor Λ_2^1

4.1.2. Dimension 3. We will start constructing the family Λ_3^1. Poisson tensors in this family will have the form $X_A \wedge \partial/\partial x_3$, with X_A a linear vector field on $\mathbb{R}^2$. Taking A into its real canonical Jordan form we will obtain a list of solvable Lie algebras that will exahust the classification presented in [Pa76].

(i) $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$:

with $\lambda_1\lambda_2 \neq 0$. If $\lambda_1\lambda_2 > 0$, the Poisson tensor can be brought to the form

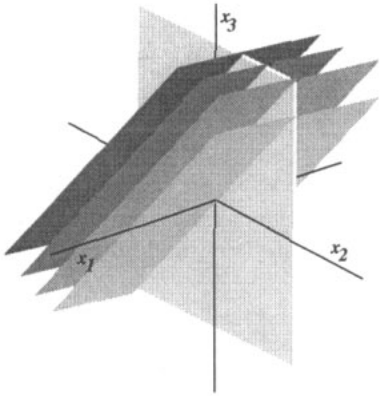

Figure 3. Symplectic leaves of the Poisson tensor Λ_3^2 with $a = 1$, $b = 0$.

$A_{3,5}^a = (x_1\partial/\partial x_1 + ax_2\partial/\partial x_2) \wedge \partial/\partial x_3$, $a > 0$. (The notation $A_{i,j}^*$ is taken from [Pa76].)

If $\lambda_1\lambda_2 < 0$, then the Poisson tensor has the form $A_{3,5}^a = (x_1\partial/\partial x_1 + ax_2\partial/\partial x_2) \wedge \partial/\partial x_3$, $a < 0$. If $a = -1$, we obtain $A_{3,5}^{-1} = A_{3,4}$.

(ii) $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, $\lambda \neq 0$:

Then the Poisson tensor can be brought to the form $A_{3,3} = (x_1\partial/\partial x_1 + x_2\partial/\partial x_2) \wedge \partial/\partial x_3$, the book algebra already discussed.

(iii) $A = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix}$:

If $\lambda \neq 0$, then the Poisson tensor acquire the form $A_{3,2} = (x_1\partial/\partial x_1 + (x_1 + x_2)\partial/\partial x_2) \wedge \partial/\partial x_3$. If $\lambda = 0$, the previous transformation does not exist, and the normal form for the Poisson tensor becomes $A_{3,1} = x_1\partial/\partial x_2 \wedge \partial/\partial x_3$. The algebra $A_{3,1}$ is nil-potent and plays an important role in the classification of higher-dimensional algebras.

(iv) $A = \begin{pmatrix} 0 & \nu \\ -\nu & 0 \end{pmatrix}$, $\nu \neq 0$:

The canonical form of the Poisson tensor is $A_{3,6} = (x_1\partial/\partial x_2 - x_2\partial/\partial x_1) \wedge \partial/\partial x_3$.

(v) $A = \begin{pmatrix} \mu & \nu \\ -\nu & \mu \end{pmatrix}$, $\mu\nu \neq 0$:

The canonical form of the Poisson tensor is $A_{3,7}^{\mu} = \mu(x_1\partial/\partial x_1 + x_2\partial/\partial x_2) \wedge \partial/\partial x_3 + A_{3,6}$.

The family Λ_3^2 (see figure 3) will be constructed in $\mathbb{R}^3$ using the Lie algebra Λ_2^1 in $\mathbb{R}^2$. We get

$$\Lambda_3^2(A) = \Lambda_2^1 + X_A \wedge \frac{\partial}{\partial x_3}$$

where $X_A = a_{11}x_1\partial/\partial x_1 + a_{12}x_1\partial/\partial x_2 + a_{21}x_2\partial/\partial x_1 + a_{22}x_2\partial/\partial x_2$, is a derivation of Λ_2^1, i.e.

$$\mathcal{L}_{X_A}\Lambda_2^1 = 0\,.$$

Because of the discussion in subsection 3.2, we conclude that all derivations of Λ_2^1 are *inner. In fact, solving the previous equation we obtain*

$$X_A = x_1\left(a\frac{\partial}{\partial x_1} + b\frac{\partial}{\partial x_2}\right) \qquad a, b \in \mathbb{R}$$

and $X_A = -bX_1 + aX_2$, where X_1, X_2 are the Poisson vector fields with Hamiltonians x_1 and x_2, respectively. Thus,

$$\Lambda_3^2 = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + x_1 \left(a \frac{\partial}{\partial x_1} + b \frac{\partial}{\partial x_2} \right) \wedge \frac{\partial}{\partial x_3} . \tag{20}$$

The Poisson tensors in the family Λ_3^2 given by (20) are degenerate, with rank 2 if $x_1 \neq 0$, and with a linear Casimir function

$$C = bx_1 - ax_2 + x_3 .$$

Symplectic leaves of this family are the planes $C =$ constant (see figure 3). Inner derivations are generated by the Poisson vector fields

$$X_1 = x_1 \frac{\partial}{\partial x_2} + ax_1 \frac{\partial}{\partial x_3} \qquad X_2 = bx_1 \frac{\partial}{\partial x_3} - x_1 \frac{\partial}{\partial x_1} \qquad X_3 = -ax_1 \frac{\partial}{\partial x_1} - bx_1 \frac{\partial}{\partial x_2} \tag{21}$$

and the image of Λ_3^2 is spanned by the vector fields X_1, X_2, X_3.

The discussion in subsection 3.2 allows as to conclude that there is an outer derivation of the form CX_a where X_a is a automorphism of Λ_3^2. The condition $\mathcal{L}_{X_a} \Lambda_3^2 = 0$ implies that X_a has the form $\alpha \partial / \partial x_2 + \beta \partial / \partial x_3$. Then a generic derivation will have the form

$$X_A = mX_1 + nX_2 + pX_3 + (bx_1 - ax_2 + x_3) \left(\alpha \frac{\partial}{\partial x_2} + \beta \frac{\partial}{\partial x_3} \right) .$$

Notice that a linear change of coordinates on $\mathbb{R}^2$ allows to choose $b = 0$, and after a rescaling of x_3 we can take $a = 1$. This solvable Lie algebra corresponds again to the nil-potent Lie algebra $A_{3,1}$.

4.1.3. Dimension 4. We shall start the construction of four-dimensional solvable Lie algebras by computing the family Λ_4^1. The Poisson tensors in this family have the form

$$\Lambda_4^1(A) = X_A \wedge \frac{\partial}{\partial x_4}$$

with A a linear map on $\mathbb{R}^3$. We can construct the following families of Poisson tensors attending to real Jordan normal forms for A.

(i) $A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$:

If $\lambda_1 \lambda_2 \lambda_3 \neq 0$, then the Poisson tensor can be taken into the normal form $A_{4,5}^{a,b} = (x_1 \partial/\partial x_1 + ax_2 \partial/\partial x_2 + bx_3 \partial/\partial x_3) \wedge \partial/\partial x_4$. If $\lambda_1 = 0$, the canonical form is then $(x_2 \partial/\partial x_2 + bx_3 \partial/\partial x_3) \wedge \partial/\partial x_4$. If $b = -1$, we obtain $A_{4,8}$.

(ii) $A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix}$, $\lambda_2 \neq 0 \neq \lambda_1$:

This tensor can be taken into the form $A_{4,2}^a = (ax_1 \partial/\partial x_1 + (x_1 + x_2)\partial/\partial x_2 + x_3 \partial/\partial x_3) \wedge \partial/\partial x_4$.

(iii) $A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$, $\lambda \neq 0$:

Then after rescaling we obtain the canonical form $A_{4,4} = ((x_1 + x_2)\partial/\partial x_1 + (x_2 + x_3)\partial/\partial x_2 x_3 \partial/\partial x_3) \wedge \partial/\partial x_4$. If $\lambda = 0$, then the canonical form is $A_{4,1} = (x_2 \partial/\partial x_1 + x_3 \partial/\partial x_2) \wedge \partial/\partial x_4$.

(iv) $A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & \nu \\ 0 & -\nu & \mu \end{pmatrix}$:

The corresponding Poisson tensor defines the family $A_{4,6}^{\mu,\lambda}$.

To construct the family Λ_4^2, we must compute the derivations of the Lie algebras in the family Λ_3^1. For instance, if we compute the derivations of the Lie algebra $A_{3,1}$, we obtain the general form

$$A = \begin{pmatrix} \alpha+\beta & \gamma & \delta \\ 0 & \alpha & \rho \\ 0 & \sigma & \beta \end{pmatrix}$$

where we identify a two-dimensional subalgebra of inner derivations $\gamma x_1 \partial/\partial x_2 + \delta x_1 \partial/\partial x_3$, a two-dimensional subspace of parallel outer derivations $\rho x_2 \partial/\partial x_3 + \sigma x_3 \partial/\partial x_2$, and a two-dimensional subspace of (nonparallel) outer derivations $(\alpha+\beta) x_1 \partial/\partial x_1 + \alpha x_2 \partial/\partial x_2 + \beta x_3 \partial/\partial x_3$ (see also figure 4).

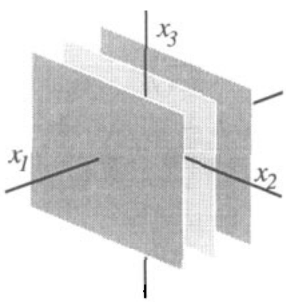

Figure 4. Symplectic leaves of the nil-potent Lie algebra $A_{3,1}$.

The list of canonical forms corresponding to these derivations are given by:

(i) $A = \begin{pmatrix} \alpha+\beta & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$, $\beta \neq 0$:

Then, after rescaling, the canonical form is given by $A_{4,9}^a = ((1+a)x_1\partial/\partial x_1 + x_2 \partial/\partial x_2 + a x_3 \partial/\partial x_3) \wedge \partial/\partial x_4$.

(ii) $A = \begin{pmatrix} \alpha+\beta & 0 & 0 \\ 0 & \alpha & \rho \\ 0 & 0 & \beta \end{pmatrix}$, $\alpha \neq 0$:

The canonical form is now given by $A_{4,7}^a = (a x_1 \partial/\partial x_1 + (x_2 + x_3)\partial/\partial x_2 + x_3 \partial/\partial x_3) \wedge \partial/\partial x_4$. If $a = 2$, we obtain $A_{4,8}$.

(iii) $A = \begin{pmatrix} 2\alpha & 0 & 0 \\ 0 & \alpha & \rho \\ 0 & \sigma & \alpha \end{pmatrix}$, $\alpha \neq 0$:

If $\rho = 1 = -\sigma$, we obtain the Poisson tensor $A_{4,11}^\alpha = (2\alpha x_1 \partial/\partial x_1 + (\alpha x_2 - x_3)\partial/\partial x_2 + (x_2 + \alpha x_3)\partial/\partial x_3) \wedge \partial/\partial x_4$. If $\alpha = 0$, we get $A_{4,10}$ which is the well known oscillator algebra.

Finally we should mention that extending the book algebra $A_{3,3}$ by the derivation $x_1 \partial/\partial x_2 - x_2 \partial/\partial x_1$, we get another element on the family Λ_4^2, $A_{4,12} = A_{3,3} + (x_1 \partial/\partial x_2 - x_2 \partial/\partial x_1) \wedge \partial/\partial x_4$. These families reproduce the list of solvable algebras in [Pa76].

5. Liouville geometry of Poisson manifolds

5.1. Poisson–Liouville geometry

The choice of a volume form Ω_0 on the vector space L^*, introduces an additional structure in the analysis of the Poisson tensor Λ_L associated to the Lie algebra L. In fact the situation is more general, and it is interesting to first consider the situation on an arbitrary orientable manifold P. If the manifold P is orientable, a volume form Ω can be chosen and a isomorphism $\Psi: V \mapsto \Psi_V = i_V\Omega$ is defined among the algebra of multivector fields $V(P)$ and the algebra of forms $\bigwedge(P)$. Notice that the image of an homogeneous multivector field V of degree k is a form Ψ_V of degree $\dim P - k$ [Ko85]. This isomorphism depends on the choice of the volume form; if Ω is replaced by $f\Omega$ the corresponding isomorphisms are related by $\Psi'_V = f\Psi_V$. We will call Ω the Liouville form of the theory.

If (P, Λ) is a Poisson structure, a simple computation shows that

$$i_{X_f}\Omega = \mathrm{d}f \wedge i_\Lambda \Omega \tag{22}$$

for any Poisson vector field $X_f = -\Lambda(\mathrm{d}f)$. Then, if θ is a closed 1-form defining the locally Poisson vector field $X_\theta = -\Lambda(\theta)$, using (22) it is easy to show that

$$i_{X_\theta}\Omega = \theta \wedge \Psi_\Lambda\,. \tag{23}$$

It is important to remark that locally Poisson vector fields X_θ need not be Liouville locally, i.e. divergenceless, with respect to the Liouville form Ω (see [Ma81, Ta94] for a thorough discussion on Nambu dynamics). In fact a simple computation shows that

$$\mathcal{L}_{X_\theta}\Omega = \mathrm{d}(i_{X_\theta}\Omega) = \mathrm{d}(\theta \wedge i_\Lambda\Omega) = -\theta \wedge \mathrm{d}(i_\Lambda\Omega) = -\theta \wedge \mathrm{d}\Psi_\Lambda\,.$$

Thus, locally Poisson vector fields with respect to Λ *will be locally Liouville with respect to* Ω iff $\mathrm{d}\Psi_\Lambda = 0$.

Definition 2. A couple (Λ, Ω), where Λ is a Poisson bi-vector and Ω a volume form on the manifold P will be called a Poisson–Liouville structure if $\mathrm{d}(i_\Lambda\Omega) = 0$.

Proposition 3. If L is a perfect Lie algebra, i.e. $[L, L] = L$, then (Λ_L, Ω_0) defines a Poisson–Liouville structure on L^*.

Proof. Let X_f be an arbitrary linear Poisson vector field on L^*, i.e. f is a linear function on L^*. Then, it is clear that $\mathrm{div}(X_f)$ is constant because

$$\mathcal{L}_{X_f}\Omega_0 = \mathrm{div}(X_f)\Omega_0$$

and $\mathrm{div}(X_f)$ must be of degree zero. Then,

$$\mathcal{L}_{[X_f, X_g]}\Omega_0 = \mathcal{L}_{X_f}(\mathrm{div}(X_g)\,\Omega_0) - \mathcal{L}_{X_g}(\mathrm{div}(X_f)\,\Omega_0) = 0\,.$$

But $[L, L] = L$ by assumption and $[X_f, X_g] = X_{\{f,g\}_L}$, thus any linear Poisson vector field X_h will be of the form $[X_f, X_g]$ for some linear f, g, and the conclusion follows. □

Perfect Lie algebras form a subclass of Poisson–Liouville structures. We will try to understand the structure of Poisson tensors according to this property, i.e. we will try to find if it is possible to decompose a Poisson tensor in a part which is compatible with a Liouville structure and a part, easy to characterize, which is not. We will proceed performing the analysis in a general background in order to make some of the constructions more transparent.

First of all it is relevant to notice that the differential complex defined on the algebra of forms by the exterior differential d can be transported to the algebra of multivector fields $V(P)$ by means of Ψ as

$$\Psi(\mathrm{D}(V)) = \mathrm{d}\Psi(V) \tag{24}$$

i.e. $\mathrm{D} = \Psi^{-1} \circ \mathrm{d} \circ \Psi$. It is clear that $\mathrm{D}^2 = 0$ but D is not a derivation on the algebra of multivector fields $(V(P), \wedge)$. In fact, the following formula is satisfied [Gr93, Ko85]:

$$\mathrm{D}(U \wedge V) = \mathrm{D}(U) \wedge V + (-1)^k U \wedge \mathrm{D}(V) + [U, V] \tag{25}$$

for $U \in V^k(P)$, $V \in V(P)$. The homology operator D will be called the divergence operator.

Chosing a bi-vector Λ, from (25) we obtain

$$\mathrm{D}(\Lambda \wedge \Lambda) - 2\Lambda \wedge \mathrm{D}(\Lambda) = [\Lambda, \Lambda] \tag{26}$$

thus, Λ defines a Poisson structure iff

$$\mathrm{D}(\Lambda^2) = 2\Lambda \wedge \mathrm{D}(\Lambda) \tag{27}$$

or, equivalently,

$$2i_\Lambda \mathrm{d}\Psi_\Lambda = \mathrm{d}\Psi_{\Lambda^2} . \tag{28}$$

Sufficient conditions for the Jacobi identity, (27) or (28), are $\mathrm{D}(\Lambda) = 0$ and $\mathrm{D}(\Lambda^2) = 0$. Notice that $\mathrm{D}(\Lambda) = 0$ is equivalent to (Λ, Ω) to define a Poisson–Liouville structure on P. We will also say in this situation that the Poisson bracket defined by Λ, is Ω-closed or closed for short. Similarly, Λ will be said to be exact if Ψ_Λ is exact.

If (Λ, Ω) is a Poisson–Liouville structure, and X_f is a Poisson vector field with respect to Λ, the generating $(n-2)$-form with respect to the Liouville form is given by $f\Psi_\Lambda$, in fact,

$$\mathrm{d}(f\Psi_\Lambda) = \mathrm{d}f \wedge \Psi_\Lambda = i_{X_f}\Omega .$$

In this sense, the geometry of Poisson brackets which are Liouville with respect to some volume form can be studied from the dual setting of Liouville structures, i.e. in the realm of forms. Questions concerning Poisson vector fields can be translated to Liouville vector fields and so on.

If we are given a Poisson bi-vector Λ, and we choose an arbitrary volume form Ω, any other volume form Ω' will be related to the last one by a nowhere vanishing function f, i.e. $\Omega' = f\Omega$. Thus, $i_\Lambda \Omega' = i_\Lambda f\Omega = f i_\Lambda \Omega$ and, $\mathrm{d}(i_\Lambda\Omega') = \mathrm{d}f i_\Lambda \Omega + f\mathrm{d}(i_\Lambda\Omega)$. Unfortunately, the equation

$$\mathrm{d}f \wedge i_\Lambda\Omega + f\mathrm{d}(i_\Lambda\Omega) = 0$$

has no solution for f, in general.

Again, the decomposition problem outlined at the beginning of this paragraph, can be rephrased in this context as follows: given Λ and Ω we will be looking for a decomposition of Λ in a D-closed part, Λ_0, and a non-closed part, i.e. a part Λ_0 which is compatible with the Liouville structure, and that is responsible for the non-closedness of the Poisson bracket.

In the following paragraphs we will collect some results that will be useful in finding an answer to this decomposition problem. Given Ψ_Λ, define the vector field X_Λ by $i_{X_\Lambda}\Omega = \mathrm{d}\Psi_\Lambda$ or, in other words,

$$X_\Lambda = D(\Lambda) . \tag{29}$$

Then,

$$i_{X_\Lambda}\mathrm{d}\Psi_\Lambda = 0 \tag{30}$$

and it follows easily that

$$\mathrm{D}(X_\Lambda) = 0\,. \tag{31}$$

It is simply to show that $i_{X_\Lambda}\Psi_\Lambda$ is exact. In fact,

$$i_{X_\Lambda}\Psi_\Lambda = i_{X_\Lambda} i_\Lambda \Omega = i_\Lambda i_{X_\Lambda}\Omega = i_\Lambda \mathrm{d}\Psi_\Lambda$$

and using the Jacobi identity, (28), we obtain

$$i_{X_\Lambda}\Psi_\Lambda = \tfrac{1}{2}\mathrm{d}\Psi_{\Lambda^2}\,. \tag{32}$$

We conclude,

Lemma 4. $\mathcal{L}_{X_\Lambda}\Psi_\Lambda = 0$.

Proof. We have to compute

$$\mathcal{L}_{X_\Lambda}\Psi_\Lambda = i_{X_\Lambda}\mathrm{d}\Psi_\Lambda + \mathrm{d}(i_{X_\Lambda}\Psi_\Lambda)\,.$$

But from (30) the first term on the right-hand side of the previous equation vanishes, and because of (32), the second term in the RHS vanishes equally. □

It is evident from the definition of X_Λ that it is a Liouville vector field

$$\mathcal{L}_{X_\Lambda}\Omega = 0\,. \tag{33}$$

It is also simple to verify, that X_Λ is a derivation.

Lemma 5. $\mathcal{L}_{X_\Lambda}\Lambda = 0$.

Proof. In fact

$$\mathcal{L}_{X_\Lambda} i_\Lambda \Omega = \Psi(\mathcal{L}_{X_\Lambda}\Lambda) + i_\Lambda \mathcal{L}_{X_\Lambda}\Omega = \Psi(\mathcal{L}_{X_\Lambda}\Lambda)$$

but from lemma 4, we get that the left-hand side in the previous formula is zero and the result follows. □

5.2. *Decomposable Poisson structures*

The results in the previous section are general for orientable Poisson manifolds. In this section we will introduce new objects which do not necessarily exist on arbitrary Poisson manifolds.

Definition 3. We will say that the Poisson tensor Λ of the Poisson manifold P is Ω-decomposable if it is closed or, if $\mathrm{d}\Psi_\Lambda \neq 0$, there exists a closed 1-form θ such that

$$\theta \wedge \mathrm{d}\Psi_\Lambda = \Omega\,. \tag{34}$$

For an arbitrary Λ, θ will exist only on a dense open submanifold of P. In what follows we will assume that θ is defined on the whole space P.

From the definition of X_Λ we will obtain,

$$\mathrm{d}\Psi_\Lambda = i_{X_\Lambda}\Omega = i_{X_\Lambda}(\theta \wedge \mathrm{d}\Psi_\Lambda) = \langle X_\Lambda, \theta\rangle\, \mathrm{d}\Psi_\Lambda$$

where we have used (30). Thus,

$$\langle X_\Lambda, \theta\rangle = 1\,. \tag{35}$$

Even if the 1-form θ exists, it is not uniquely defined. In fact, if θ' is another closed 1-form satisfying (34) it is clear that the closed 1-form $\beta = \theta' - \theta$ is a constant of the motion for X_Λ. In particular if $\beta = df$, then $X_\Lambda(f) = 0$.

The locally Poisson vector field $X_\theta = -\Lambda(\theta)$ satisfies the formulae expressed in the following lemmas.

Lemma 6. $\mathcal{L}_{X_\theta}\theta = 0$.

Proof. A direct computation shows that

$$\mathcal{L}_{X_\theta}\theta = i_{X_\theta}\mathrm{d}\theta + \mathrm{d}(i_{X_\theta}\theta) = \mathrm{d}(-i_{\Lambda(\theta)}\theta) = -\mathrm{d}\Lambda(\theta,\theta) = 0\,. \qquad \square$$

Lemma 7. $\mathcal{L}_{X_\theta}\Omega = -\Omega$.

Proof. Again, computing directly the Lie derivative we get

$$\mathcal{L}_{X_\theta}\Omega = \mathrm{d}(i_{X_\theta}\Omega) = \mathrm{d}(\theta \wedge \Psi_\Lambda) = -\theta \wedge \mathrm{d}\Psi_\Lambda = -\theta \wedge i_{X_\Lambda}\Omega = -\langle X_\Lambda, \theta\rangle\Omega = -\Omega$$

where we have made use of the closedness of θ and (35). $\square$

Lemma 8. $[X_\theta, X_\Lambda] = 0$.

Proof.

$$i_{[X_\Lambda, X_\theta]}\Omega = \mathcal{L}_{X_\Lambda} i_{X_\theta}\Omega - i_{X_\theta}\mathcal{L}_{X_\Lambda}\Omega = \mathcal{L}_{X_\Lambda}(\theta \wedge \Psi_\Lambda)$$

where we have used (33). Then,

$$i_{[X_\Lambda, X_\theta]}\Omega = \mathcal{L}_{X_\Lambda}\theta \wedge \Psi_\Lambda + \theta \wedge \mathcal{L}_{X_\Lambda}\Psi_\Lambda = \mathrm{d}(i_{X_\Lambda}\theta) \wedge \Psi_\Lambda = 0$$

where we have used lemma 4 and (35). $\square$

Finally, we notice that

Lemma 9. $D(X_\theta) = -1$.

Proof. Clearly,

$$i_{\mathrm{D}(X_\theta)}\Omega = \mathrm{d}\Psi_{X_\theta} = \mathrm{d}(\theta \wedge \Psi_\Lambda) = -\theta \wedge \mathrm{d}\Psi_\Lambda\,.$$

But $\theta \wedge i_{X_\Lambda}\Omega = \langle X_\Lambda, \theta\rangle\Omega = \Omega$, and the conclusion follows. $\square$

5.3. Decomposition theorem

Theorem 1. Let (P, Λ) be a Poisson manifold with Liouville form Ω. If Λ is Ω-decomposable it has a decomposition

$$\Lambda = \Lambda_0 + X_\theta \wedge X_\Lambda$$

where Λ_0 is D-closed, and $\mathrm{D}(X_\theta \wedge X_\Lambda) = -X_\Lambda$.

Proof. Let us compute directly $D(\Lambda_0)$. We obtain

$$\mathrm{D}(\Lambda_0) = \mathrm{D}(\Lambda) - \mathrm{D}(X_\theta \wedge X_\Lambda) = X_\Lambda - [X_\theta, X_\Lambda] + \mathrm{D}(X_\theta)X_\Lambda - \mathrm{D}(X_\Lambda)X_\theta$$

but because of lemmas 8 and 9, we get,

$$\mathrm{D}(\Lambda_0) = X_\Lambda - X_\Lambda = 0$$

then, Λ_0 is D-closed.

Because lemma 8, it is simple to show that $X_\theta \wedge X_\Lambda$ defines a Poisson bracket. In fact

$$[X_\theta \wedge X_\Lambda, X_\theta \wedge X_\Lambda] = 2X_\theta \wedge X_\Lambda \wedge [X_\theta, X_\Lambda] = 0\,.$$

The compatibility of Λ_0 and $X_\theta \wedge X_\Lambda$ is guaranteed by $\mathcal{L}_{X_\theta}\Lambda = 0$ and lemmas 5 and 8. In fact, because X_θ is locally Poisson,

$$0 = \mathcal{L}_{X_\theta}\Lambda = \mathcal{L}_{X_\theta}\Lambda_0\,.$$

Then $[X_\theta \wedge X_\Lambda, \Lambda_0] = 0$. $\square$

Finally, we should mention that the locally Poisson vector field corresponding to θ with respect to the Poisson structure Λ_0 is zero,

$$\Lambda_0(\theta) = \Lambda(\theta) - (X_\theta \wedge X_\Lambda)(\theta) = -X_\theta + X_\theta = 0. \qquad (36)$$

We will apply the previous results to the linear Poisson tensor Λ_L defined by a Lie algebra L. First we must notice that Ψ_{Λ_L} will be homogeneous of degree $n-1$, where n is the dimension of L^*. Hence X_{Λ_L} will be a constant vector field, i.e. of degree -1. In fact, a short computation shows that

$$X_{\Lambda_L} = (-1)^n \operatorname{Tr} C_i \frac{\partial}{\partial x_i}\,. \qquad (37)$$

Notice that if L is unimodular Λ_L is Poisson–Liouville.

We can state first the following fact.

Proposition 4. The Poisson tensor Λ_L associated with a Lie algebra L, is Poisson–Liouville iff Poisson vector fields associated with linear functions are divergenceless.

Proof. The necessity is obvious from the definition. On the contrary, if $\mathrm{d}f \wedge \mathrm{d}\Psi_\Lambda = 0$, for any linear function then $\mathrm{d}\Psi_\Lambda = 0$, and Λ_L is Poisson–Liouville. □

Theorem 2. Every Lie algebra defines a Poisson tensor Λ_L, which can be decomposed as

$$\Lambda_L = \Lambda_0 + X_\theta \wedge X_{\Lambda_L}$$

where Λ_0 is a Poisson–Liouville tensor.

Proof. Because Ψ_{Λ_L} is homogeneous of degree $n-1$, then $\mathrm{d}\Psi_{\Lambda_L}$ is a $(n-1)$–form homogeneous of degree $n-1$. That implies that $\mathrm{d}\Psi_{\Lambda_L}$ has the general form

$$\mathrm{d}\Psi_{\Lambda_L} = a_{i_1 \ldots i_{n-1}} \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_{n-1}}$$

with $a_{i_1 \ldots i_{n-1}}$ constants. It this form is non-zero, it implies that for some $i_1, \ldots, i_{n-1}$, the previous coefficient does not vanish. Take $\theta = a^{-1}_{i_1 \ldots i_{n-1}} \mathrm{d}x_j$ with j not in the list $i_1, \ldots, i_{n-1}$. Then (L^*, Λ_L) is Ω-decomposable in the sense of definition 3 and we apply theorem 1. □

The 1-form θ is linear, $\mathcal{L}_\Delta \theta = \theta$, but because it is closed, it must be exact as shown in the proof before. Thus the vector field X_θ is linear, and consequently it defines a linear map A, $X_\theta = X_A$. It is clear from the definition that $\mathrm{D}(X_A) = \operatorname{Tr} A$. Thus lemma 9 implies that $\operatorname{Tr} A = -1$. In fact,

$$i_{D(X_\theta)}\Omega = \mathrm{d}(i_{X_\theta}\Omega) = \mathrm{d}\left(\sum_{i=1}^{n-1} (-1)^i A_i{}^j x_j \mathrm{d}x_1 \wedge \cdots \widehat{\mathrm{d}x_i} \cdots \wedge \mathrm{d}x_n\right) = \operatorname{Tr} A\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n\,.$$

Notice that it is enough to have $\operatorname{Tr} A \neq 0$ because we can always rescale the coordinates to obtain the desired value for $\operatorname{Tr} A$.

6. Semi-simple Lie algebras

We have shown that semi-simple Lie algebras are Poisson–Liouville. The extension procedure discussed in section 3 will tell us how to construct Poisson tensors corresponding to semi-simple Lie algebras out of simple ones. Thus, we will concentrate our attention on linear Poisson tensors corresponding to simple Lie algebras.

Notice that a function $f \in C^\infty(L^*)$ is in the centre iff $\{f, h\}_L = 0$ for every h, but this implies that $\Lambda_L(\mathrm{d}f \wedge \mathrm{d}h) = 0$, i.e. $\mathrm{d}f$ is in the kernel of Λ_L, or equivalently $X_f = 0$. Thus, because of (22) f is a Casimir for $\{\cdot,\cdot\}_L$ iff $\mathrm{d}f \wedge \Psi_{\Lambda_L} = 0$. More, generally, a Casimir

1-form α will be any 1-form such that $\alpha \wedge \Psi_{\Lambda_L} = 0$. Of course, $i_{X_h}\alpha = 0$ for any function h. Because of the homomorphism property (7), when the symplectic leaves of Λ_L have codimension one, α has an integrating factor, i.e. locally α is associated with a Casimir function. The family of forms such that $i_\alpha \Lambda = 0$ constitute a differential ideal generated by local Casimir functions.

It is clear that for some Lie algebras, in addition to an invariant volume, there is also an invariant pseudometric tensor defined on L^*, say $g = g^{ij} \mathrm{d}x_i \otimes \mathrm{d}x_j$, with respect to Poisson vector fields associated with linear functions on L^*.

Definition 4. We shall say that Λ_L is dynamically compatible with a pseudometric tensor g if Poisson vector fields associated with linear functions are Killing vector fields for g.

For instance, Abelian Lie algebras are dynamically compatible with any metric tensor.

Proposition 5. Central extensions of simple Lie algebras admit compatible metric tensors.

Proof. If L is a simple Lie algebra with Poisson structure Λ_L a central extension by $\mathbb{R}^k$ will be described by a form

$$\tilde{\Psi}_\Lambda = \mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}y_k \wedge (\Psi_\Lambda + c)$$

where c represents the $(n-2)$-form defined by the cocycle defining the extension, and $(y_1, \ldots, y_k)$ are linear coordinates in $\mathbb{R}^k$. The volume form we are choosing on $\mathbb{R}^k \oplus L^*$ is $\tilde{\Omega}_0 = \mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}y_k \wedge \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$. If $g_L = g^{ij} \mathrm{d}x_i \otimes \mathrm{d}x_j$ corresponds to the Cartan–Killing metric tensor on L we have a compatible metric tensor given by

$$\tilde{g}_L = f\pi^* g_L + A^{\alpha\beta} \mathrm{d}y_\alpha \otimes \mathrm{d}y_\beta$$

with $\mathrm{d}A^{\alpha\beta} \wedge \tilde{\Psi}_L = 0$, $\mathrm{d}f \wedge \tilde{\Psi}_L = 0$, $f \neq 0$, $\det A^{\alpha\beta} \neq 0$, and $\pi\colon \mathbb{R}^k \oplus L^* \to L^*$ the natural projection. □

It should be noticed that when $k = 0$ we have compatible metric tensors for a simple Lie algebra by considering the Cartan–Killing metric and multiplying it by any non-zero function of the Casimirs. When a compatible pseudometric tensor is available we can construct a scalar product on the tensor algebra and we can construct Casimir functions. Therefore $\langle \Lambda, \Lambda \rangle$ will be an invariant function. Then $\mathrm{d}\langle \Lambda, \Lambda \rangle$ is a Casimir 1-form, and thus

$$\mathrm{d}\langle \Lambda, \Lambda \rangle \wedge \Psi_\Lambda = 0 .$$

In general, for any higher power $\langle \Lambda \wedge \Lambda, \Lambda \wedge \Lambda \rangle, \ldots, \langle \Lambda^k, \Lambda^k \rangle$, we get Casimir functions. By adapting the proof given by Racah [Ra50] we can show that these generate all Casimir functions. In addition, for any invariant tensor T we can construct $\langle T, T \rangle$ which is an invariant function (this provides a method for computing invariants alternative to [Pa76, Pe94]).

Proposition 6. Any Λ_L dynamically compatible with a covariant pseudometric tensor g of degree-2 (or contravariant of degree-2) is a subalgebra of a simple Lie algebra.

Proof. If $g = g^{ij} \mathrm{d}x_i \otimes \mathrm{d}x_j$, then linear transformations preserving g are elements of $so(n, k)$. Therefore, $X_f = -\Lambda_L(\mathrm{d}f)$ are going to be elements in $so(n, k)$. Because they close on a Lie algebra, they will close a subalgebra of $so(n, k)$. □

Notice that the integral submanifolds of $\mathrm{Im}\,\Lambda_L$ are contained into orbits of subalgebras of $so(n, k)$ in L^*.

In addition to the map Ψ it is clear that we can construct a Hodge star operator $*\colon \bigwedge^{n-k}(L^*) \to \bigwedge^k(L^*)$ associated to g. The 2-form $*\Psi_{\Lambda_L}$ is the 2-form that defines a 2-form symplectic on the symplectic leaves of Λ_L (coadjoint orbits on L^*). It should be noticed that $*\Psi_{\Lambda_L}$ need not be closed on L^*. It is only closed along the leaves, i.e. $d\Psi_\Lambda = 0$ and,

$$d(*\Psi_{\Lambda_L})(X_f, X_g, X_h) = 0$$

for any choice of functions f, g, h on L^*.

7. Classification of low-dimensional real Lie algebras

We will consider the classification problem of Lie algebras in low dimensions. We will discuss the classification up to four dimensions to indicate how the ideas discussed along the paper apply in these simple situations. Because of the decompostion theorem for Lie algebras, theorem 2, it suffices to look for closed Lie algebras and extensions of the form discussed in section 3, and we can use the machinery developed in subsection 5.2.

7.1. Dimension 2.

Let Λ be a Poisson tensor of degree -1 in a two-dimensional space with linear coordinates x_1, x_2, and volume form chosen to be $\Omega_0 = dx_1 \wedge dx_2$, then Ψ_Λ is a linear function, and if Λ is closed, $d\Psi_\Lambda = 0$ implies that $\Psi_\Lambda = 0$ too. Thus $\Lambda = 0$, and the only Liouville–Poisson algebras in two dimensions are Abelian.

If Λ is now a general bi-vector, it must be a monomial of the form

$$\Lambda = X_A \wedge \frac{\partial}{\partial x_2}$$

where $X_A = ax_1 \partial/\partial x_1$. Computing Ψ_Λ we get

$$\Psi_\Lambda = ax_1$$

(that will be closed only if $a = 0$). If $a \neq 0$, rescaling the coordinates we can choose it such that $\operatorname{Tr} A = -1$, and we obtain the algebra $[e_1, e_2] = -e_1$. In this case $\Lambda_0 = 0$.

7.2. Dimension 3.

We shall now consider a linear bi-vector Λ in a three-dimensional space with coordinates x_1, x_2, x_3 and volume form $\Omega_0 = dx_1 \wedge dx_2 \wedge dx_3$. Then Ψ_Λ is a 1-form of degree 2 and $\mathcal{L}_\Delta \Psi_\Lambda = 2\Psi_\Lambda$. Furthermore, $\Lambda^2 = 0$ in dimension 3, and then if $d\Psi_\Lambda = 0$, Λ automatically defines a Lie algebra structure. Consequently, all closed 1-forms of degree 2 define Liouville–Poisson algebras, hence all 1-forms admitting a local integrating factor define a Lie algebra. We should mention of course that the Lie bracket with Poisson tensor Λ may reduce to Λ_0, i.e. $D(\Lambda) = 0$ from the start.

Let Λ be a Poisson tensor of degree -1 with corresponding 1-form Ψ_Λ of degree 2,

$$\Psi_\Lambda = A_{ij} x_i \, dx_j$$

that can be decomposed as

$$\Psi_\Lambda = \tfrac{1}{2}(A_{ij} - A_{ji}) x_i \, dx_j + \tfrac{1}{2}(A_{ij} + A_{ji}) x_i \, dx_j \, .$$

Defining $a^k = \epsilon^{ijk} A_{ij}$ we get,

$$\Psi_\Lambda = \tfrac{1}{2} a^k \epsilon_{ijk} (x_i \, dx_j - x_j \, dx_i) + \tfrac{1}{2} d(A_{ij} x_i x_j) \, .$$

Because $d\Psi_\Lambda = a^k \epsilon_{ijk} \, dx_i \wedge dx_j$, we conclude that $X_\Lambda = a^k \partial/\partial x_k = X_a$. The 1-form θ can be chosen to be dx_3 and $X_a = \partial/\partial x_3$, then

$$\Psi_\Lambda = \tfrac{1}{2}(x_1 \, dx_2 - x_2 \, dx_1) + \tfrac{1}{2} d(A_{ij} x_i x_j)$$

and imposing $\Psi_\Lambda \wedge \mathrm{d}\Psi_\Lambda = 0$ we find $i, j \in \{1, 2\}$.

Therefore either,

$$\Psi_\Lambda = \tfrac{1}{2}(x_1\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1) + \tfrac{1}{2}\mathrm{d}(A_{ij}x_ix_j) \qquad i, j \in \{1, 2\}$$

or,

$$\Psi_\Lambda = \tfrac{1}{2}\mathrm{d}(A_{ij}x_ix_j)\,.$$

Because A_{ij} is symmetric in two dimensions, we can bring it to normal form by using a rotation matrix which preserves the form of $x_1\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1$, we find

$$\Psi_\Lambda = \tfrac{1}{2}(y_1\,\mathrm{d}y_2 - y_2\,\mathrm{d}y_1) + \tfrac{1}{2}\mathrm{d}\big(ay_1^2 + by_2^2\big)$$

or,

$$\Psi_\Lambda = \tfrac{1}{2}\mathrm{d}\big(a_1y_1^2 + a_2y_2^2 + a_3y_3^2\big)\,.$$

The algebra $A_{3,1}$ is closed with associated 1-form $-\mathrm{d}x_1^2$. The algebra $A_{3,2}$ is not closed and its associated 2-form is $2(x_1\mathrm{d}x_2 - x_2\mathrm{d}x_1) - \mathrm{d}x_1^2$. The algebra $A_{3,3}$ is not closed and has the form $x_1\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1$. The algebra $A_{3,4}$ is closed with $\Psi_\Lambda = d(x_1x_2)$. On the contrary the algebras $A_{3,5}^a$ are not closed with $\Psi_\Lambda = x_1\,\mathrm{d}x_2 - ax_2\,\mathrm{d}x_1$. The algebra $A_{3,6}$ is closed with form $-\frac{1}{2}\mathrm{d}\big(x_1^2 + x_2^2\big)$. The algebra $A_{3,7}^a$ is not closed with 1-form $a(x_1\mathrm{d}x_2 - x_2\mathrm{d}x_1) - \frac{1}{2}\mathrm{d}\big(x_1^2 + x_2^2\big)$. For the semi-simple algebras $A_{3,8}$, $A_{3,9}$, the respective forms $-\mathrm{d}(x_2^2 + x_1x_3)$ and $-\frac{1}{2}\mathrm{d}\big(x_1^2 + x_2^2 + x_3^2\big)$ are closed.

7.3. Dimension 4.

We shall consider now four-dimensional Lie algebras. The linear coordinates will be denoted by x_1, x_2, x_3, x_4 with volume form $\Omega_0 = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4$.

All simple algebras have only inner derivations, therefore the extension procedure with them provides only central extensions. It is also easy to show that Poisson–Liouville algebras in four dimensions are necessarily of the type $\Psi_\Lambda \wedge \Psi_\Lambda = 0$, with $\mathrm{d}\Psi_\Lambda = 0$. From it we have $\Psi_\Lambda = \mathrm{d}A_\Lambda$. In fact, if $\mathrm{D}(\Lambda) = 0$, this implies that $\mathrm{D}(\Lambda^2) = 0$, or equivalently, $\mathrm{d}\Psi_{\Lambda^2} = 0$. But, Ψ_{Λ^2} is a function of degree 2, whose differential vanishes, thus it must be constant, then zero. But then, $0 = \Psi_{\Lambda^2}$ that implies $\Psi_\Lambda \wedge \Psi_\Lambda = 0$.

By adapting an argument from electromagnetism it is not difficult to show that either $A_\Lambda = A_i(x_1, x_2, x_3)\,\mathrm{d}x^i$, where $i \in \{1, 2, 3\}$ or $A_\Lambda = f(x_1, x_2, x_3, x_4)\,\mathrm{d}x_4$, with A_i and f quadratic functions of their arguments.

We notice that in the second case we have $\mathrm{d}f \wedge \mathrm{d}x_4 = \Psi_\Lambda$, therefore the associated Lie algebra L cannot be perfect, i.e. it cannot satisfy $[L, L] = L$. In the first case $A_\Lambda \wedge \Psi_\Lambda = A_\Lambda \wedge \mathrm{d}A_\Lambda = 0$, i.e. A_Λ is a Casimir 1-form. The associated bi-vector field has the form $X_B \wedge \partial\partial x_4$ with

$$X_B = \epsilon_{ijk}\left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}\right)\frac{\partial}{\partial x_k}$$

therefore again the algebra cannot be perfect. Thus the four-dimensional case is absorbed in the listing of solvable algebras discussed already in section 4.

For higher dimensions the analysis becomes more involved but it is still feasible. For instance, it is quite easy to find the results by Turkowsky [Tu88] and to generalize to solvable Lie algebras the analysis of

[Mu63] for six-dimensional nil-potent algebras.

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References

[Al93] Alekseevsky D V and Perelomov A M 1993 Poisson brackets on simple Lie algebras and symplectic Lie algebras *Preprint* CERN-TH.6984/93

[Gr93] Grabowski J, Marmo G and Perelomov A 1993 Poisson structures: towards a classification *Mod. Phys. Lett.* A **8** 1719–33

[Ja62] Jacobson N 1962 *Lie Algebras* (New York: Dover)

[Ko85] Koszul J-L 1985 Crochet de Schouten–Nijenhuis et cohomologie Astérisque *Soc. Math. de France hors série* 257–71

[Li77] Lichnerowicz A 1977 Les variétés de Poisson et leurs algèbres de Lie associées *J. Diff. Geom.* **12** 253–300

[Li88] Lie S and Engel F 1888 *Theorie der Transformationsgruppen I–III* 2nd edn (1970) (Teubner)

[Li92] Liu Z-J and Xu P 1992 On Quadratic Poisson Structures *Lett. Math. Phys.* **26** 33–42

[Ma81] Marmo G, E J Saletan, Simoni A and Zaccaria F 1981 Liouville dynamics and Poisson brackets *J. Math. Phys.* **22** 835–42

[Mg78] Magri F 1978 A simple model of the integrable Hamiltonian equation *J. Math. Phys.* **19** 1156–62

[Mu63] Mubarakzyanov G M 1963 *Izv. Vysshikh Uchebn. Zavedenii Mat.* **1** 114; **3** 99; 4 104

[Nd94] Ndogmo J C and Winternitz P 1994 Solvable Lie algebras with Abelian nilradicals *J. Phys. A: Math. Gen.* **27** 405–23

[Ni55] Nijenhuis A 1955 *Indag. Math.* **17** 390–403

[Pa76] Patera J, Sharp R T and Winternitz P 1976 Invariants of real low dimension Lie algebras *J. Math. Phys.* **17** 986–94

[Pe94] Pecina-Cruz J N 1994 An algorithm to calculate the invariants of any Lie algebra *J. Math. Phys.* **35** 3146–62

[Ra50] Racah G 1950 Sulla caratterizzazione delle rappresentazioni irriducibili dei gruppi semisimplici di Lie Rendiconti Acad. dei Lincei *Sc. Fis. Mat. Nat.* **8** 108–12

[Sh55] Schenkman E 1955 On the derivation algebra and the holomorph of a nil-potent Lie algebra *Memoirs Am. Math. Soc.* **14** 15–22

[Sc53] Schouten J A 1954 On the differential operators of first order in tensor calculus *Conv. Int. Geom. Diff.* (Rome 1953) ed Cremonese pp 1–7

[Ta94] Takhtajan L 1994 On foundations of the generalized Nambu Mechanics *Commun. Math. Phys.* **160** 295–315

[Tu74] Tulczyjew W M 1974 Poisson brackets for canonical manifolds *Bull. Acad. Pol. Sci., Ser. Sci. Mat. Astro.* **22** 931–4

[Tu88] Turkowski P 1988 Low dimensional real Lie algebras *J. Math. Phys.* **29** 1239–44

[We83] Weinstein A 1983 The local structure of Poisson manifolds *J. Diff. Geom.* **18** 523–57